Extension Lesson
Conics and Foci

NAME:

Objectives

- Determine the foci from the equation of an ellipse or hyperbola.
- Derive the equations of an ellipse and hyperbola from the foci.

Extension of:

This lesson extends the learning from Unit 3.

Explore

Conics and Foci

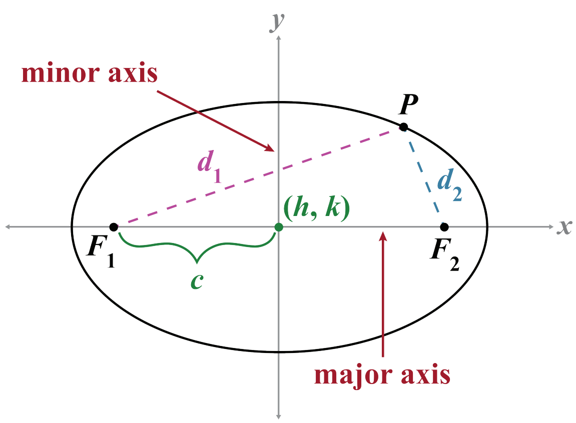
- **Focal points, or foci,** F_1 and F_2, are two fixed points on the major axis of an ellipse or hyperbola.
  - For an ellipse, the sum of the distances from any point, P, on an ellipse to the foci is a constant value: $d_1 + d_2 = \text{constant}$
  - For a hyperbola, the absolute value of the difference of the distance from any point, P, on a hyperbola to the foci is a constant value: $|d_1 - d_2| = \text{constant}$
- **Focal length**, c, is the distance from the center (h, k) of an ellipse or hyperbola to one focal point.
- The midpoint formula can be used to locate the center if the foci are given.
- When $a > 0$ and $b > 0$:
  - The major axis (ellipse) or the transverse axis (hyperbola) is a segment with length $2a$.
  - The minor axis (ellipse) or the conjugate axis (hyperbola) is a segment with length $2b$.

In this lesson, a will always represent the major or transverse axis.

- You must first determine if the major or the transverse axis is vertical or horizontal to work with focal points.

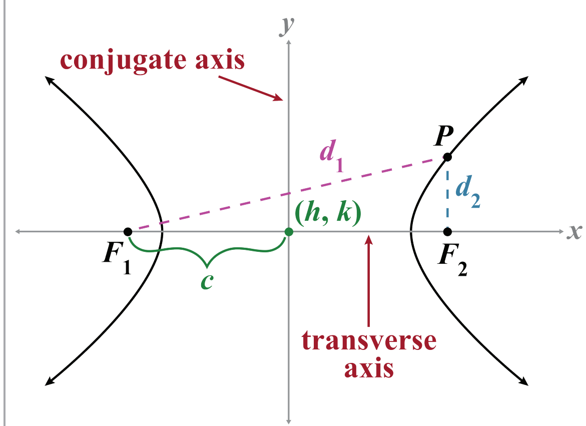
<table>
<tr><th></th><th>Horizontal Ellipse</th><th>Horizontal Hyperbola</th></tr>
<tr><td>Horizontal major axis with center (h, k)</td><td>

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

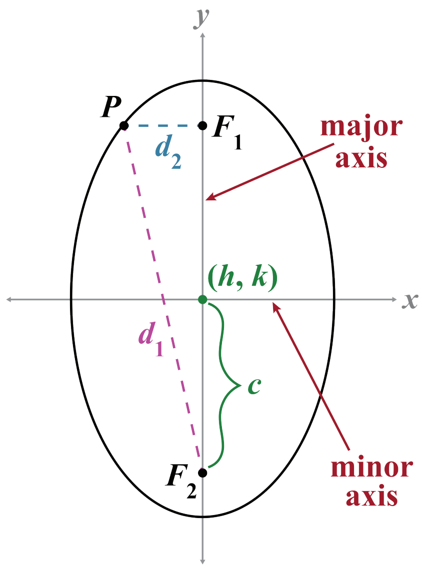
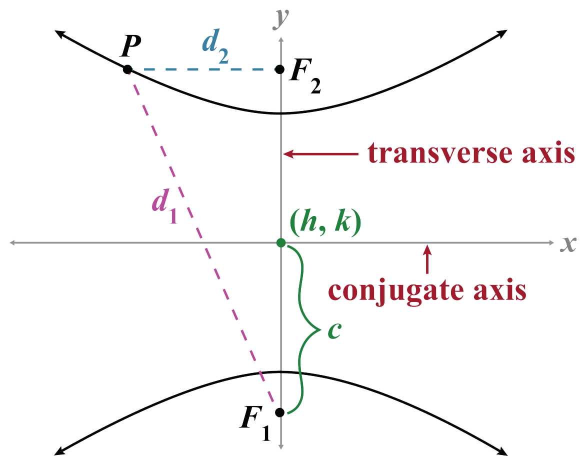


</td><td>

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$



</td></tr>
<tr><td>Horizontal foci</td><td>Have the same y-coordinate: $(h \pm c,\ k)$</td><td>Have the same y-coordinate: $(h \pm c,\ k)$</td></tr>
<tr><td>Axes</td><td>Major axis: $2a$
Minor axis: $2b$
$a > b$, always</td><td>Transverse axis: $2a$
Conjugate axis: $2b$
The transverse axis is represented by the term with x in the numerator.</td></tr>
<tr><td>Focal length</td><td>$(\text{major})^2-(\text{minor})^2=(\text{focal})^2$
or
$a^2-b^2=c^2$</td><td>$(\text{transverse})^2+(\text{conjugate})^2=(\text{focal})^2$
or
$a^2+b^2=c^2$</td></tr>
</table>

	Vertical Ellipse	Vertical Hyperbola
Major vertical axis with center (*h*, *k*)	$\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$ 	$\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$ 
Vertical foci	Have the same x-coordinate: $(h,\ k\pm c)$	Have the same x-coordinate: $(h,\ k\pm c)$
Axes	Major axis: $2a$ Minor axis: $2b$ $a>b$, always	Transverse axis: $2a$ Conjugate axis: $2b$ The transverse axis is represented by the term with y in the numerator.
Focal length	$(\text{major})^2-(\text{minor})^2=(\text{focal})^2$ or $a^2-b^2=c^2$	$(\text{transverse})^2+(\text{conjugate})^2=(\text{focal})^2$ or $a^2+b^2=c^2$

Example 1

State whether the equation is a vertical or horizontal ellipse or hyperbola. Then determine the foci.

$$\frac{(y-2)^2}{144} - \frac{(x+7)^2}{25} = 1$$

Vertical hyperbola

Solve for c:

$a^2 = 144$

$b^2 = 25$

$a^2 + b^2 = c^2$

$144 + 25 = c^2$

$c^2 = 169$

$c = 13$

Explain

- y is positive; subtraction between terms: vertical hyperbola
- Solve for c with $a^2 + b^2 = c^2$
- Find foci with $(h,\ k \pm c)$

Find the foci:

$(h,\ k): (-7,\ 2)$

$(h,\ k \pm c)$

$(-7,\ 2+13) = (-7,\ 15)$

$(-7,\ 2-13) = (-7,\ -11)$

Example 2

Determine the equation of the ellipse in standard form when the length of its vertical minor axis is 14 units and the focal points are at $(-30,\ -8)$ and $(18,\ -8)$.

Center:

$$\left(\frac{-30+18}{2},\ \frac{-8+(-8)}{2}\right) = (-6,\ -8)$$

Minor axis (b):

$2b = 14$

$b = 7$

Focal length (c):

$(-6,\ -8), (-30,\ -8)$

$c = \sqrt{(-30-(-6))^2 + (-8-(-8))^2} = \sqrt{576}$

$c = 24$

Explain

- y-coordinates are equal, ellipse is horizontal
- Midpoint formula to find the center
- Distance formula to find the focal length
- Solve for a
- Write the equation of a horizontal ellipse

Find a:

$a^2 - b^2 = c^2$

$a^2 - 7^2 = 24^2$

$a^2 - 49 = 576$

$\sqrt{a^2} = \sqrt{625}$

$a = 25$

Equation: $\frac{(x+6)^2}{625} + \frac{(y+8)^2}{49} = 1$

Example 3

Determine the equation of the hyperbola in standard form with vertices at $(1,\ 5)$ and $(1,\ -1)$ and focal points at $(1,\ 7)$ and $(1,\ -3)$.

Center:

$$\left(\frac{1+1}{2}, \frac{5+-1}{2}\right)=(1, 2)$$

Focal length (c):

$(1, 7), (1, 2)$

$$c=\sqrt{(1-1)^2+(7-2)^2}=\sqrt{5^2}$$

$$c=5$$

Find a:

$(1, 2), (1, -1)$

$$a=\sqrt{(1-1)^2+(2-(-1))^2}=\sqrt{3^2}$$

$$a=3$$

Equation: $\dfrac{(y-2)^2}{9}-\dfrac{(x-1)^2}{16}=1$

Explain

- x-coordinates are equal, vertical hyperbola
- Center: midpoint
- Focal length: distance formula
- Transverse axis (major): distance formula
- Find b
- Write equation

Find b:

$$a^2+b^2=c^2$$

$$3^2+b^2=5^2$$

$$9+b^2=25$$

$$\sqrt{b^2}=\sqrt{16}$$

$$b=4$$

Practice

Complete problems on a separate sheet of paper.

Identify the ellipse or hyperbola as horizontal or vertical. Then find the focal points.

1) $\frac{(x+5)^2}{289} + \frac{(y-6)^2}{64} = 1$

2) $\frac{x^2}{36} - \frac{(y+2)^2}{64} = 1$

3) $\frac{(x+1)^2}{144} + \frac{(y-8)^2}{169} = 1$

4) $\frac{x^2}{289} + \frac{y^2}{64} = 1$

5) $\frac{y^2}{225} - \frac{x^2}{64} = 1$

6) $\frac{(y-17)^2}{49} - \frac{(x+15)^2}{576} = 1$

Write the equation for the named conic in standard form.

7) A horizontal ellipse with vertices at $(15, 0)$ and $(-15, 0)$ and focal length of 12 units.

8) A hyperbola with a horizontal transverse axis of 10 and foci at $(0, 0)$ and $(26, 0)$.

9) A horizontal ellipse with vertices at $(17, 0)$ and $(-17, 0)$ and a focal length of 8 units.

10) An ellipse with focal points at $(-2, 5)$ and $(-2, -11)$ and co-vertices at $(4, -3)$ and $(-8, -3)$.

11) A hyperbola with a vertical transverse axis of 48 and foci at $(2, 1)$ and $(2, 51)$.

12) An ellipse with focal points at $(2, 8)$ and $(14, 8)$ and vertices at $(0, 8)$ and $(16, 8)$.

13) A hyperbola with a horizontal transverse length of 12 and foci at $(-7, 0)$ and $(7, 0)$.

14) A hyperbola with vertices at $(5, 2)$ and $(1, 2)$ and focal points at $(7, 2)$ and $(-1, 2)$.