Extension Lesson
Geometric Sequences and Series

NAME:

Objectives

- Find the common ratio of a geometric sequence.
- Write a formula to determine the n^{th} term of a geometric sequence.
- Find the sum of a finite geometric sequence.

Extension of:

This lesson extends the learning from Unit 5.

Explore

Geometric Sequence

- A sequence is an ordered list of numbers that contains a pattern.
- A geometric sequence is an ordered list of numbers in which each term after a_1 is found by multiplying the previous term by the common ratio, *r*.
- The common ratio is the number that each term is multiplied by to find the next term in the sequence.
- A finite list can be written with variables as: $a_1, a_1r, a_1r^2, a_1r^3, \ldots, a_1r^{n-1}$
- An infinite list can be written with variables as: $a_1, a_1r, a_1r^2, a_1r^3, \ldots$
- To find successive terms in a geometric sequence when a_1 and *r* are known: $a_{n+1} = a_n \cdot r$
- To find the n^{th} term in a geometric sequence, use the formula: $a_n = \left(a_1\right) \cdot r^{n-1}$, where *n* is any natural number.
- To find *r*, divide successive terms in a geometric sequence: $r = \dfrac{a_{n+1}}{a_n}$

Example 1

For the geometric sequence, $\{128, 32, 8, 2, \ldots\}$, complete the following:

A) Determine the common ratio.

Implement

$$r = \frac{a_{n+1}}{a_n} = \frac{a_4}{a_3} = \frac{2}{8} = \frac{1}{4}$$

Explain

- Choose any two consecutive terms
- Substitute terms into: $r = \frac{a_{n+1}}{a_n}$
- Simplify

B) Determine the next term in the geometric sequence.

Implement

Next term = 5th

$$a_n = 2,\ r = \frac{1}{4}$$

$$a_{n+1} = a_n \cdot r$$

$$a_5 = 2\left(\frac{1}{4}\right) = \frac{2}{4}$$

$$a_5 = \frac{1}{2}$$

Explain

- Define terms
- Substitute terms into: $a_{n+1} = a_n \cdot r$
- Solve

C) Find the ninth term in the sequence.

Implement

$$n = 9,\ a_1 = 128,\ r = \frac{1}{4}$$

$$a_n = \left(a_1\right) \cdot r^{n-1}$$

$$a_9 = (128)\left(\frac{1}{4}\right)^{9-1}$$

$$a_9 = 128\left(\frac{1}{4}\right)^8 = \frac{1}{512}$$

Explain

- Define terms
- Substitute terms into: $a_n = \left(a_1\right) \cdot r^{n-1}$
- Solve

Example 2

Find the first term, a_1, whose fourth and fifth terms are –54 and 162, respectively.

Plan

Solve for r

Substitute known values, a_4 and r, into $a_n = \left(a_1\right) \cdot r^{n-1}$

Solve for a_1

Implement

$a_4 = -54,\ a_5 = 162$

$$r = \frac{a_{n+1}}{a_n} = \frac{a_5}{a_4} = \frac{162}{-54} = -3$$

$$a_n = \left(a_1\right) \cdot r^{n-1}$$

$$a_4 = a_1(-3)^{4-1}$$

$$-54 = a_1(-3)^3$$

$$-54 = a_1(-27)$$

$$a_1 = \frac{-54}{-27} = 2$$

Geometric Series

- A series is the sum of the terms in a sequence.
- A geometric series is the SUM of the terms in a finite geometric sequence.
- Remember, a finite sequence has a defined end value.
- $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ \ldots + a_1 r^{n-1}$ where S_n is the sum of the first n-terms.

The sum of the first n-terms of a finite geometric series is:	In which:
$S_n = \frac{a_1 \cdot \left(1 - r^n\right)}{(1-r)}$ and $S_n = \frac{a_1 - r a_n}{(1-r)}$	a_1 is the first term r is the common ratio $r \neq 1$

- For example, in the geometric sequence, $\{-80, 20, -5, 1.25\}$, the sum can be calculated in a few different ways because there are not many values in the sequence.

  Option 1: Add the terms.

  $$-80+20+(-5)+1.25=-63.75$$

  Option 2: Substitute values into either of the geometric series formulas.

  $$S_4=\frac{a_1-\left(-\frac{1}{4}\right)a_4}{\left(1-\left(-\frac{1}{4}\right)\right)}=\frac{-80+\left(\frac{1}{4}\right)1.25}{\left(1+\frac{1}{4}\right)}=-63.75$$

  Option 3: Calculate the sum of a series.

  $$S_4=-80+(-80)\left(\frac{1}{4}\right)+(-80)\left(\frac{1}{4}\right)^2+(-80)\left(\frac{1}{4}\right)^3+(-80)\left(\frac{1}{4}\right)^4=-63.75$$

- The sum of a geometric series can be represented using the uppercase sigma, Σ, in sigma notation.

  last value of k ↘ n

  $$\sum_{k=1} f(r)^k \quad \leftarrow \text{formula for the series}$$

  first value of k ↗ 1

- If the value of k equals a number other than one, the number of terms in the series is determined by $n-k+1$.

Example 3

Leonhard decides to start saving for a used car. He saves $75 in the first month. In subsequent months, he increases the contribution by 5% every month. Calculate the total savings after 18 months.

$a_1=75$

$r=1+0.05=1.05$

$n=18$

$$S_{18}=\frac{a_1\cdot\left(1-r^{18}\right)}{(1-r)}=\frac{75\left(1-(1.05)^{18}\right)}{1-1.05}=2109.928$$

After 18 months, Leonhard will have a total of $2,109.93 saved.

Example 4

Determine the specified value.

$$\sum_{k=-2}^{5} (3)^k$$

Implement

$$a_1 = 3^{-2} = \frac{1}{9}$$

$$r = 3$$

Number of terms:

$$n - k + 1 = 5 - (-2) + 1 = 8$$

$$S_8 = \frac{\frac{1}{9}\left(1 - (3)^8\right)}{1 - 3} = \frac{1}{9}\left(\frac{1 - 3^8}{-2}\right) = \frac{3280}{9} = 364.\overline{4}$$

Explain

- Determine a_1, r, and n
- Substitute values into the geometric series formula
- Solve

Practice

Complete problems on a separate sheet of paper.

For problems 1–2, use the geometric sequence: $\{5, -10, 20, -40, \ldots\}$

1) Determine the common ratio.

2) Find a_7, a_{10}.

For problems 3–4, use $a_6 = 9$ **and** $r = \frac{1}{3}$.

3) Find a_1.

4) Write the first three terms of the sequence.

For problems 5–6, use the scenario.

A person processes 15% of a 100 milligram (100 mg) dose of a certain medication every hour (i.e., $r = 1 - 0.15 = 0.85$).

5) Determine the amount of medication left in a person's body after 3 hours.

6) After approximately how long will a person have 50 milligrams of medication remaining in their body?

For problems 7–10, find the missing value. Round to the nearest hundredth.

7) $a_1 = 0.8, \quad r = 2.1, \quad S_{12} = \_\_\_$

8) $\sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k-2}$

9) $S_8 = 1640, \quad r = 3, \quad a_1 = \_\_\_$

10) $\sum_{k=-1}^{6} \left(\frac{5}{4}\right)^{k}$

For problems 11–12, use the scenario.

Kristin saved $100 in the first month and increased the amount by 3% in subsequent months.

11) Calculate the total savings for a year.

12) If Kristin has continued the pattern of saving and currently has $8,202.32 in her account, how long has she been saving?

For problems 13–14, use the scenario.

A basketball is dropped from a height of 6 feet above the ground. Each bounce is 70% of the height of the previous bounce. After the initial drop, the remaining distances are traveled twice (up and down).

13) Determine the height of the second and third bounce.

14) After the initial drop, the remaining distances are traveled twice (up and down, or $S_{\text{bounces}} = 2\left(S_{\text{heights}}\right)$). How far did the ball travel after four bounces? (Hint: The drop is not counted twice.)

To continue, return to the Online Lesson.